

An Iterative Method for Solving Quadratic Optimal Control Problem Using Scaling Boubaker Polynomials

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Abstract:

In this paper, an iterative method was used for solving a quadratic optimal control problem (QOCP) by the aid of state parameterization technique and scaling Boubaker polynomials. Some numerical examples were added to show the applicability of the method, also a comparison with other method was presented. The process steps were illustrated by some numerical examples with graphs done by Matlab.

Keywords: Quadratic optimal control problem, State parameterization technique, Scaling Boubaker function, Iterative method.

Introduction

Optimal control represents a large field in which many researchers gave different methods of various aspects. The crucial aim for solving an optimal control problem (OCP) is to find the control variable which minimizes a given performance index while all the given constraints are satisfied. State parameterization method is one of the mostly used direct methods in solving (OCPs) by the researchers. Kafash B., Delavarkhalafi A. and Karbassi S. M., used state parameterization for solving (NOCPs) and the controlled Duffing Oscillator [1]. Ouda E. H., utilized generalized Laguerre polynomials as a basis function with the aid of state-control parameterization to find an approximate solution for (OCPs) [2].

Scaling functions are a functional dilation equation has the general following form

$$f(t) = \sum_{k=0}^N C_k f(2t - K), \quad 0 \leq t \leq 1, C_k \text{ are real or complex coefficients ... (1)}$$

With a nonzero solution, this kind of equations has been used in many fields (e.g., interpolating subdivision schemes and wavelet theory) [3]. Yousfi S. A., presented a numerical solution of Emden–Fowler equations using Legendre scaling function approximation [4]. Ouda E. and Ahmed I. utilized direct methods (state and control Parameterization) with the scaling Boubaker function for solving OCP[5].

Iterative technique was also largely used for solving OCPs in last decades, Keyanpour M. and Azizsefat M., had an iterative approach with a hybrid of perturbation and parameterized methods for this purpose [6]. Ramezanzpour H. et al., presented a new procedure depending on homotopy perturbation method with iterative technique to solve an optimal control problem of a bilinear systems [7]. Elaydi H., Jaddu H. and Wadi M., utilized Legendre scaling function with iterative technique for solving NOCP [8]. Jaddu H. and Majdalawi A., presented an iterative technique in two proceedings, firstly by parameterizing the state variables by finite length Chebyshev series [9], secondly by a finite length Legendre polynomials [10].

Eskandri M. et al., introduced a method for solving a class of nonlinear quadratic optimal control problem (NQOCP) based on variational iteration method [11]. Ramezani M., proposed a new iterative method utilizing 2nd kind Chebyshev wavelet [12]. Shihab S. and Delphi M., used the iterative technique on B-spline Bernestein polynomials [13].

Boubaker polynomials are proved to be a good tool for solving (OCPs), Samaa F. et al., used indirect method based on Boubaker polynomial [14]. Ouda E. H., deduced the operational matrices of derivative and integration and using them with the indirect method [15]. Many other researchers deal with this kind of polynomials in different proceedings. The novelty of our approach is using scaling Boubaker polynomials for solving (QOCP's), this method was proved to be efficient and accurate. A comparison was introduced to show the capability of this method with some other methods.

This paper is arranged as follows, in section2, Boubaker polynomials have been presented. In section3, scaling Boubaker polynomials were introduced. In section 4, the proposed method was presented in steps. In section 5, some numerical examples with comparison for the first example and illustrative figures were added to show the capability of this method, at the end conclusion and the references.

Boubaker polynomials

The Boubaker polynomials were established for the first by Boubaker et al. as a tool for solving heat equation inside physical model, and then it was used for solving different equations in many applications. [16]

Boubaker polynomial is introduced by the following equation [17]

$$B_n(t) = \sum_{p=0}^{\zeta(n)} \left[\frac{(n-4p)}{(n-p)} C_{n-p}^p \right] (-1)^p t^{n-2p}, \quad n = 0, 1, 2, \dots \quad \dots(2)$$

$$\text{where } C_{n-p}^p = \frac{(n-p)!}{p!(n-2p)!}, \quad \zeta(n) = \left\lfloor \frac{n}{2} \right\rfloor = \frac{2n + ((-1)^n - 1)}{4}.$$

m is the degree of Boubaker polynomials .

The first terms of Boubaker polynomials are

$$B_0(t) = 1, \quad B_1(t) = t, \quad B_2(t) = t^2 + 2, \dots$$

and the recurrence relation $B_m(t) = tB_{m-1}(t) - B_{m-2}(t)$, where $m > 2$

Scaling boubaker polynomials (SBP)

The Scaling Boubaker polynomials (SBP), can be defined as follows [18]

$$SB_{nm}(t) = \begin{cases} 2^{\frac{k}{2}} B_m(2^{k+1}t - 2n - 1) & \frac{2n-1}{2^{k+1}} \leq t \leq \frac{2n}{2^{k+1}} \\ 0 & \text{Otherwise} \end{cases} \dots(3)$$

The arguments of scaling (k, n, m, t) , k is positive integer, $n = 0, 1, 2, 3, \dots, 2k$, m is degree of Boubaker polynomials and t is the time.

Choosing $k=1$ and $m=5$. The first five terms Scaling Boubaker $SB_m(t)$ were found by using (3) to be:

$$SB_0 = \sqrt{2},$$

$$SB_1 = \sqrt{2}(4t - 1),$$

$$SB_2 = \sqrt{2}(16t^2 - 8t + 3),$$

$$SB_3 = \sqrt{2}(64t^3 - 48t^2 + 16t - 2),$$

$$SB_4 = \sqrt{2}(256t^4 - 256t^3 + 96t^2 - 16t - 1).$$

The convergence of this method with state parameterization technique has been treated in [2].

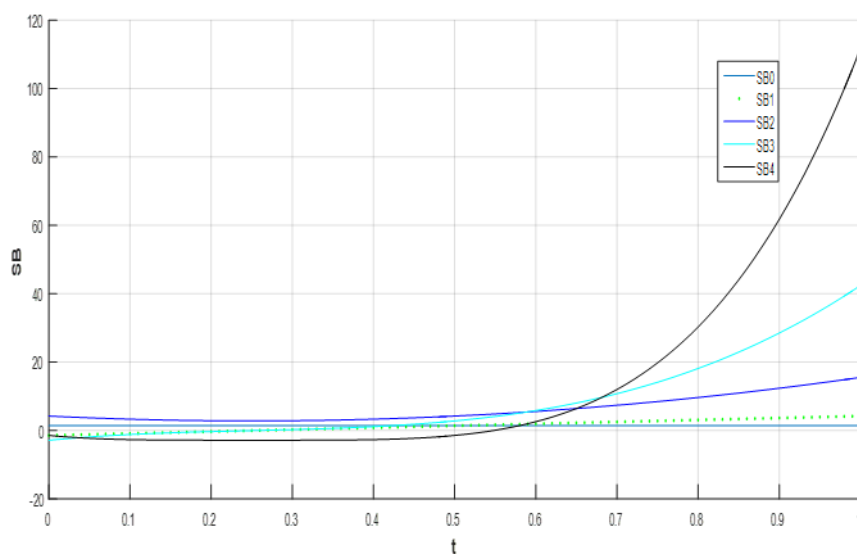


Figure 1. Scaling Boubaker polynomials (n = 4)

The proposed method

The state parameterization is based on approximating state variables by using Scaling Boubaker polynomials (SBP) with unknown coefficients as follows,

$$x(t) = \sum_{i=0}^m a_i SB_i(t) = a^T SB(t) \quad t_0 \leq t \leq t_f \dots(4)$$

where a_i 's are unknown coefficients of state, SB are the Scaling Boubaker polynomials.

$$x(t) = a_0 SB_0 + a_1 SB_1 + a_2 SB_2 + a_3 SB_3 + \dots + a_n SB_n$$

with initial condition $x(t_0) = \sum_{i=0}^4 a_i SB_i(t_0) = x_0$, where the state coefficients a_i 's must be found.

The process steps are

- For the 1st iteration, representing the unknown function x by the first three terms of the scaling Boubaker polynomial with their coefficients a_i 's on the interval [0,1] as

$$x(t) = a_0 SB_0 + a_1 SB_1 + a_2 SB_2 \dots(5)$$

- Substituting (5) in constraint equation for finding the control variable u with the unknown coefficients a_i 's.
- Using now the performance index to find J, we get a set of algebraic equations with the a_i 's, which can simply be found by the aid of the initial conditions.
- Repeating the first iteration with more terms of the scaling Boubaker polynomial for the second and third iterations, evaluating the J_i 's and compare their values with respect to J_{exact} .
- The iterative process continues until a predetermined acceptable error value.

Numerical Examples

Ex.1: Consider the following quadratic optimal control problem [19]

$$\text{Min } J = \int_0^1 (x^2 + u^2) dt$$

$$\text{with } u = \dot{x} \quad \text{and} \quad x(0) = 0, \quad x(1) = \frac{1}{2}.$$

The exact solution is

$$x_{exact}(t) = \frac{e^{(e^t - e^{-t})}}{2(e^2 - 1)}, \quad u_{exact}(t) = \frac{e^{(e^t + e^{-t})}}{2(e^2 - 1)}, \quad J_{exact} = 0.328258821374830$$

The results of the 1st Iteration are

$$x_1 = (5/44)t^2 + (17/44)t + (1/1306897830861018)$$

$$u_1 = (5/22)t + (17/44)$$

$$J_1 = 0.328598484848489$$

The results of the 2nd Iteration are
 $x_2 = (7/86)t^3 - (4/473)t^2 + (202/473)t + (1/317825433919741)$
 $u_2 = (21/86)t^2 - (8/473)t + (202/473)$
 $J_2 = 0.328259337561646$

The results of the 3rd Iteration are
 $x_3 = (21/2242)t^4 + (145/2314)t^3 + (115/41138)t^2 + (483/1136)t + (1/704329587969285)$
 $u_3 = (42/1121)t^3 + (435/2314)t^2 + (115/20569)t + (483/1136)$
 $J_3 = 0.328258830708990$

From table1, we noticed that our proposed method has less absolute error with respect to Delphi and Mehne in which power and Bernstein polynomials have been used.

Table 1. The values of cost functional J in Example 1

Iteration	J(approximate) proposed method	Absolute error for our method	Delphi Method [19]	Mehne Method [20]
1	0.328598484848489	3.39663×10^{-4}	3.379×10^{-4}	5.0×10^{-3}
2	0.328259337561646	5.16186×10^{-7}	2.1814×10^{-4}	3.4×10^{-3}
3	0.328258830708990	9.33416×10^{-9}	2.03089×10^{-4}	2.1×10^{-4}

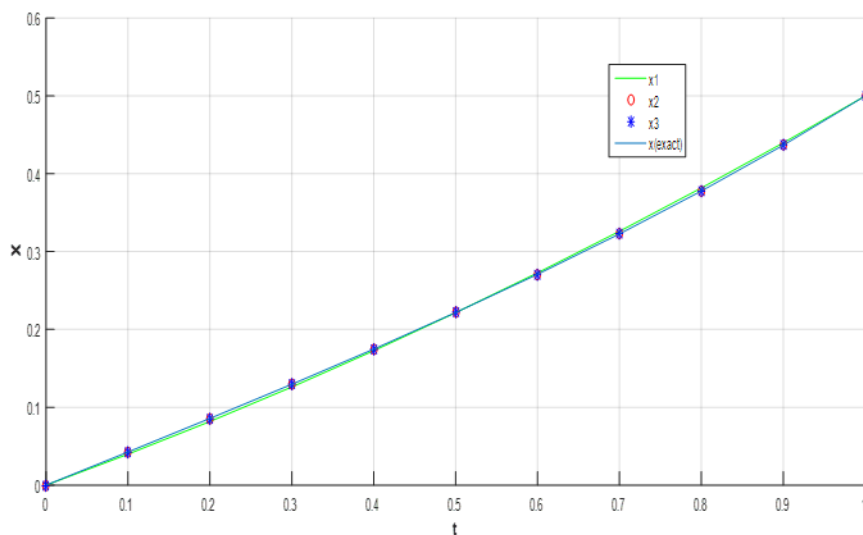


Figure 2 Example 1 - x variables

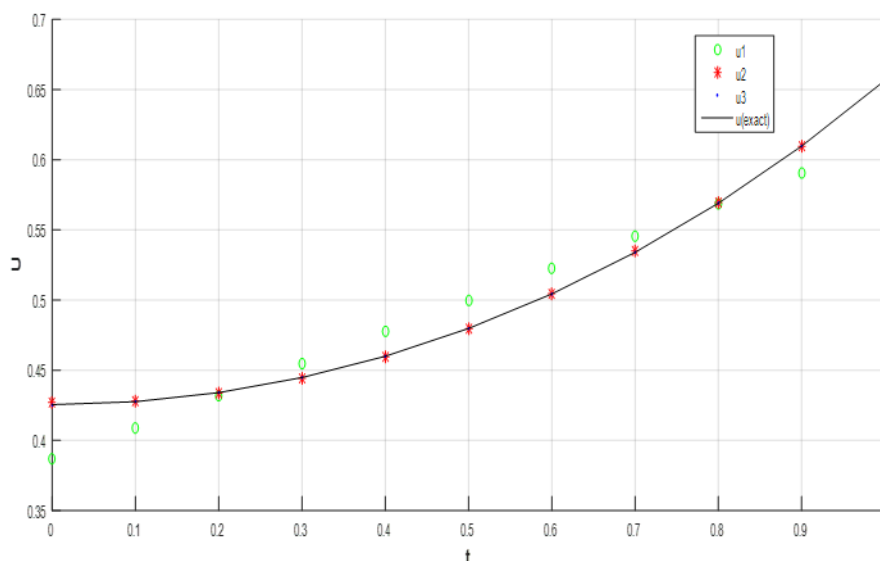


Figure 3 Example 1 – u variables

Ex.2: Consider the problem [21]

$$\text{Min } J = \frac{1}{2} \int_0^1 (x^2 + u^2) dt, \quad 0 \leq t \leq 1.$$

$$u = \dot{x} + x \text{ with } x(0) = 1$$

The exact solution is $x_{\text{exact}}(t) = Ae^{\sqrt{2}t} + (1 - A)e^{-\sqrt{2}t}$

$$u_{\text{exact}}(t) = A(\sqrt{2} + 1)e^{\sqrt{2}t} - (1 - A)(\sqrt{2} - 1)e^{-\sqrt{2}t}, \text{ where}$$

$$A = \frac{2\sqrt{2}-3}{2\sqrt{2}-(e^{\sqrt{2}})^2-3}$$

$$J_{\text{exact}}=0.192909298093169$$

The results of the 1st Iteration are

$$x_1=(100/187)t^2-(234/187)t+1$$

$$u_1=(100/187)t^2-(2/11)t-(47/187)$$

$$J_1=0.194295900178253$$

The results of the 2nd Iteration are

$$x_2=(634/2775)t^3+(1125/1283)t^2-(757/554)t+1$$

$$u_2=-(634/2775)t^3+(1965/10264)t^2+(907/2342)t-203/554$$

$$J_2=0.192931605611848$$

The results of the 3rd Iteration are

$$x_3=(113/1297)t^4-(772/1917)t^3+(1014/1033)t^2-(1128/815)t+1$$

$$u_3=(113/1297)t^4-(335/6179)t^3-(111/490)t^2+(1262/2179)t-(313/815)$$

$$J_3=0.192909445024077$$

Table 2 Numerical Results of Example 2

Iteration	J(approximate)	Absolute error
1	0.194295900178253	1.3866×10^{-3}
2	0.192931605611848	2.23075×10^{-5}
3	0.192909445024077	1.46930×10^{-7}

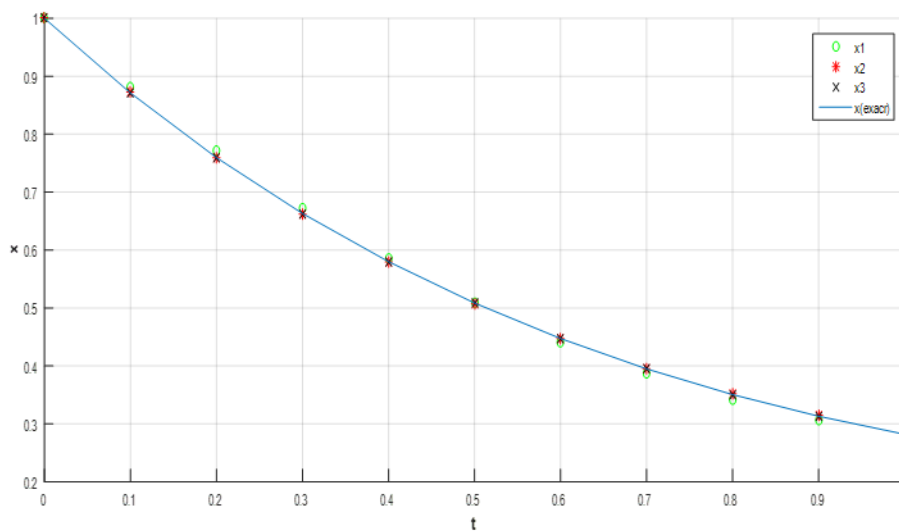


Figure 4 Example 2-x variables

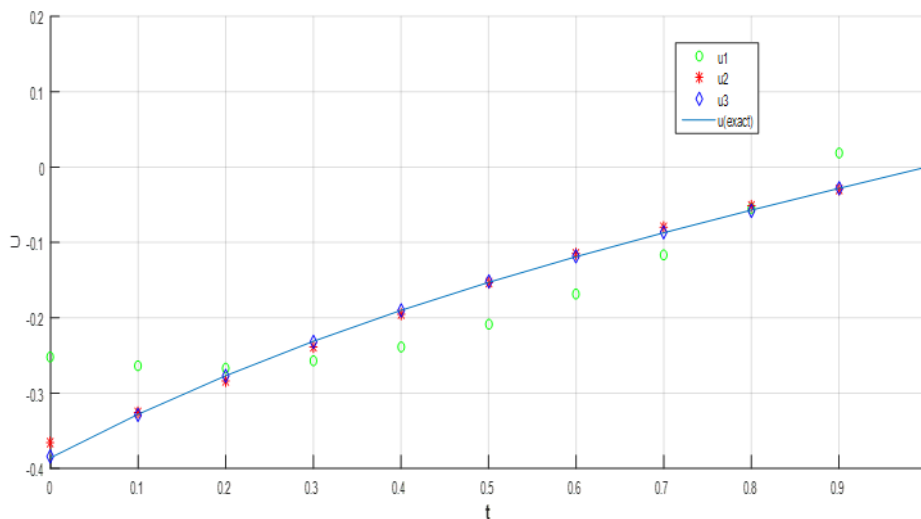


Figure 5 Example 2- u variables

Ex.3: Consider the problem [2]

$$\text{Min } J = \frac{1}{2} \int_0^1 (2x^2 + u^2) dt, \quad 0 \leq t \leq 1.$$

$$u = \dot{x} - \frac{x}{2}, \quad x(0) = 1,$$

and

$$x_{\text{exact}} = \frac{2e^{3t} + e^5}{e^{\frac{3t}{2}}(2 + e^5)}, \quad u_{\text{exact}} = \frac{2(e^{3t} - e^5)}{e^{\frac{3t}{2}}(2 + e^5)}, \quad J_{\text{exact}} = 0.864164497$$

The results of the 1st Iteration are

$$x_1 = (54/73)t^2 - (412/365)t + 1$$

$$u_1 = (746/365)t - (27/73)t^2 - (1189/730)$$

$$J_1 = 0.864726027397260$$

The results of the 2nd Iteration are

$$x_2 = -(233/1673)t^3 + (1067/1126)t^2 - (625/521)t + 1$$

$$u_2 = (233/3346)t^3 - (691/775)t^2 + (2001/802)t - (1771/1042)$$

$$J_2 = 0.864218070459266$$

The results of the 3rd Iteration are

$$x_3 = (295/2182)t^4 - (841/2053)t^3 + (1471/1325)t^2 - (919/749)t + 1$$

$$u_3 = -(266/3935)t^4 + (1741/2335)t^3 - (1586/889)t^2 + (2678/945)t - (1031/597)$$

$$J_3 = 0.864164568963970$$

Table 3 Numerical Results of Example 3

Iteration	J (approximate)	Absolute error
1	0.864726027397260	$5.61530e \times 10^{-4}$
2	0.864218070459266	$5.35734e \times 10^{-5}$
3	0.864164568963970	$7.19639e \times 10^{-8}$

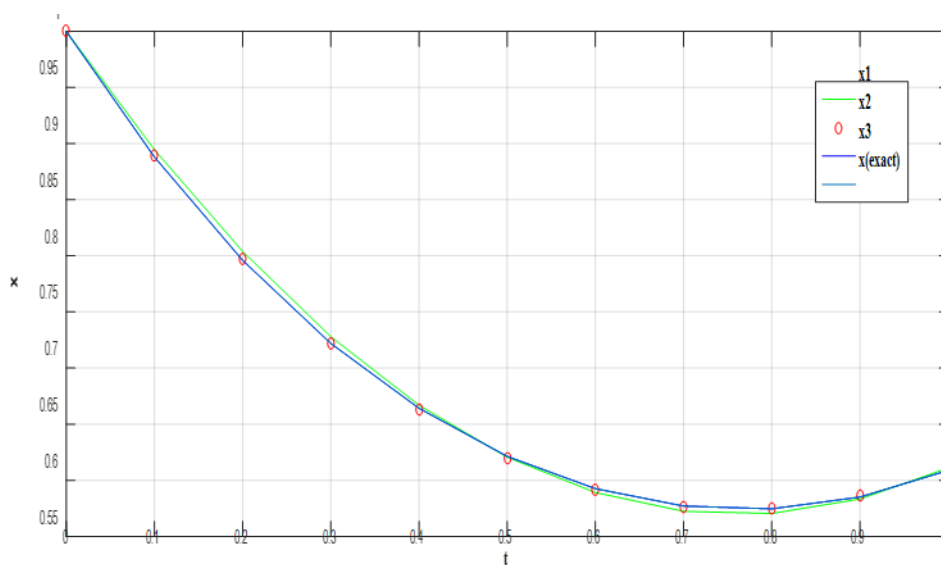


Figure 6 Example3 -x variables

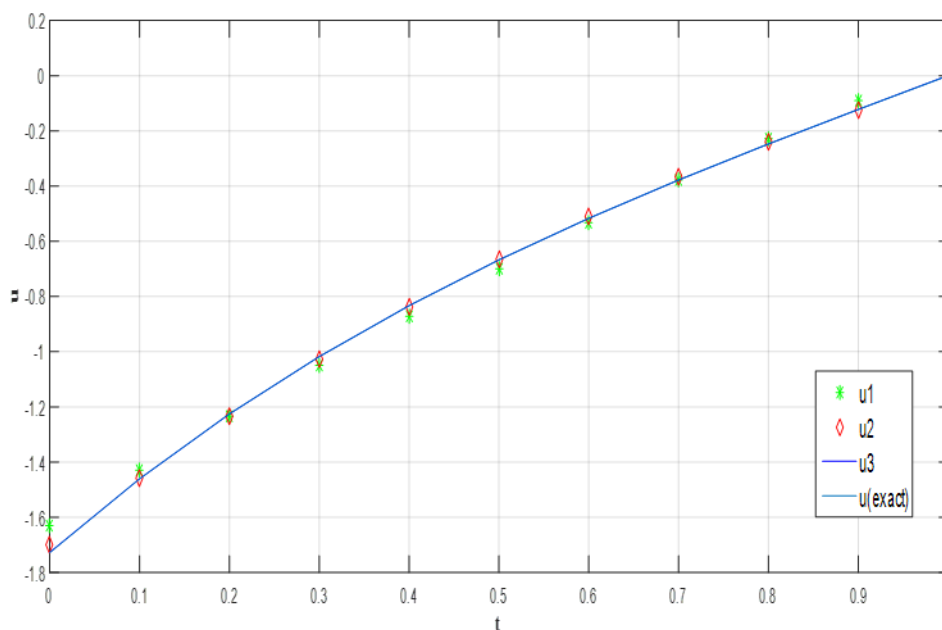


Figure 7 Example 3 - u variables

Conclusion

An iterative method with state parameterization technique using scaling Boubaker polynomials was proved to be a good tool for evaluating the optimal solution of (QOCP) by its rapid convergence and simplicity. The numerical examples show the applicability and accuracy of this method, also comparison with other methods proves its efficiency.

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