RESEARCH ARTICLE

Two criteria in Fermat infinite descent method

Anatoly A. Grinberg[1]*

[1](retired) AT&T Bell Laboratories, Murray Hill, New Jersey, USA

*Corresponding author: Anatoly A. Grinberg: anatolyfrgr@gmail.com

Abstract:

A method infinite descent is traditionally used to proof the Fermat's theorem for the special case of exponent n=4. At each step, the method sequentially generates a new Fermat's equation with one of the term being smaller than that in the preceding step. After a finite number of steps the term becomes less than one and this is taken as criterion of the insolvency of the original Fermat's equation. We show that the power of factor 2, in even parameter of Pythagoras' equation solution used in the proof, decreases by one at each step of the descent. As a result we arrive at an unsolvable equation. This is the second criterion for the descent method. Which of the two criteria is reached first depends on the parameters of the initial Pythagorean solutions chosen for the analysis.

Introduction

It is convenient to write Fermat's equation with exponent n= 4 in the form

$$(\mathbf{X^2})^2 + (\mathbf{Y^2})^2 = (\mathbf{Z^2})^2 \qquad (I.1)$$

The variables X, Y, Z are positive, pair-wise relatively prime integers, where X is even, and Y, Z are odd [1-8].

Equation (I.1) shows that the squares of these variables, $\boldsymbol{X^2}$, $\boldsymbol{Y^2}$, $\boldsymbol{Z^2}$, must coincide with the uniquely defined solutions of Pythagoras' equation, which are determined by the following expressions

$$\boldsymbol{X^2 = 2su \quad (a); \quad Y^2 = s^2 - u^2 \ (b); \quad Z^2 = s^2 + u^2 \ (c)} \qquad \text{(I.2)}$$

where $\boldsymbol{s}$ and $\boldsymbol{u}$ being the odd and the even positive parameters, respectively [4], subject to inequality s > u and (s, u)=1. It is evident that solutions for squares Y and Z, (I.2b) and (I.2c), represent Pythagorean type equations. Therefore, their solutions can also be written in the form (I.2), but with different pairs of parameters.

Despite the fact that the squares of numbers X, Y, Z are integers, the Fermat's theorem states that the square roots of these numbers cannot all be integer at the same time.

Fermat proved the theorem for exponent n=4 by the method of infinite descent. At sequential steps of the descent a new Fermat's equation arises with one of the terms smaller than that in the preceding step. After some finite number of steps the term becomes less than one and this is taken as the criterion of insolvency of Fermat' equation in integer numbers.

Below we will show that the exponent of 2 that is factor in the even parameter u, decreases by one with each step of the descent. After some number of steps the parameter becomes odd and we arrive at an unsolvable equation. This represents an second criterion for the descent method. Which criterion takes place first depends on the initial values of the parameters s and u in the Pythagorean solutions (I.2).

Proof of the theorem

Let us write even parameter u in the form

$$\boldsymbol{u = 2^{2k-1}\overline{u}} \qquad \text{(II.1)}$$

where $\boldsymbol{\overline{u}}$ is an odd number.

For the solutions of Pythagorean equations, the exponent of 2 in Eq. (II.1) can be any arbitrary positive number. However, due to Eq. (I.2a), it must be an odd number. Indeed, substituting (II.1) into (I.2a)

$$\mathbf{X^2 = 2^{2k} \, s \, \overline{u}} \qquad \text{(II.2)}$$

we se that only for even exponent of 2 can X be an integer number. Taking into account that $\mathbf{(s, \overline{u}) = 1}$, the numbers s and $\boldsymbol{\overline{u}}$ must be squares of some relatively prime odd numbers p and q, defined as

$$\mathbf{s = p^2 \ ; \ \overline{u} = q^2 \ ; \quad u = 2^{2k-1}q^2} \qquad \text{(II.3)}$$

Let us substitute (II.3) into (I.2b)

$$\mathbf{Y^2 + (2^{2k-1}\,q^2)^{\,2}) \;\; = (p^2)^2} \tag{II.4}$$

In Eq. (II.4) all terms are pair-wise relatively prime integers and its solution is

$$\mathbf{2^{2k-1}\,q^2 = 2s_1\,u_1 \;\;(a); \;\; Y = {s_1}^2 - {u_1}^2 \;\;(b); \;\; p^2 = {s_1}^2 + {u_1}^2 \;\;(c);} \tag{II.5}$$

where $(\mathbf{s_1, u_1}) = \mathbf{1}$ and $(\mathbf{s_1, p}) = \mathbf{1}$, $(\mathbf{u_1, p}) = \mathbf{1}$.

Since q and $\mathbf{s_1}$, are odd, the power of 2 in (II.5a) must be equal to (2k-2) and $\mathbf{u_1}$ can be written as

$$\mathbf{u_1 = 2^{2k-2}\,\bar{u}_1} \tag{II.6}$$

where $\mathbf{\bar{u}_1}$ is an odd number. Therefore, similarly to (II.3), it follows from (II.5a) that

$$\mathbf{s_1 = {p_1}^2 ; \;\; \bar{u}_1 = {q_1}^2 ; \;\; u_1 = 2^{2k-2}{q_1}^2} \tag{II.7}$$

where $\mathbf{p_1, q_1}$ are odd and $(\mathbf{p_1, q_1}) = \mathbf{1}$..

A comparison of Eqs. (II.3) and (II.7) shows that the power of 2 in $\mathbf{u_1}$ is less than the power of 2 in u by one. After a finite number of steps of the descent the corresponding even parameter u becomes odd. Then all three terms of (II.5c)-type equation become odd. Clearly, such numbers cannot satisfy the equation. This constitutes the second criterion for the descent method. Either this criterion or the traditional Fermat criterion will determine termination of the descent procedure. The number of steps required until parameter u becomes odd is equal to the power of 2 in expression (II.1). Which of the two criteria will be effective is determined by the values of the parameters s and u in the initial Pythagorean solutions (I.2). Both are required for the consistent application of the descent procedure.

Appendix

In this section, we follow the procedure of descent for two additional steps, in order to demonstrate the decrease in the exponent of 2 by two more units. Substituting (II.7) into (II.5c) we obtain an equation

$$\left(2^{2k-2}{q_1}^2\right)^2 \; + (p_1^2)^2 \; = \mathbf{p}^2 \tag{III.1}$$

with the solution

$$2^{2k-2}{q_1}^2 = 2s_2 u_2 \;\;(a); \;\; p_1^2 = s_2^2 - u_2^2 \;(b); \;\; p \; = s_2^2 + u_2^2 \;(c) \tag{III.2}$$

where $(s_2, u_2) = 1$, $(s_2, p_1) = 1$, $(u_2, p_1) = 1$

Since q_1 and s_2 odd, it follows from Eq.(III.2a) that u_2 has the form

$$u_2 = 2^{2k-3}\,\bar{u}_2 \qquad\qquad (III.3)$$

where $\bar{u}_2$ is odd.

From (III.2a), similar (II.7), we have

$$s_2 = p_2^2 \quad ; \quad \bar{u}_2 = q_2^2 \ ; \ u_2 = 2^{2k-3}\,q_2^2 \qquad\qquad (III.4)$$

The equation

$$\left(2^{2k-3}q_2{}^2\right)^2 \ + \ p_1^2 \ = \left(p_2^2\right)^2 \qquad\qquad (III.5)$$

obtained by substituting (III.4) into (III.2b), has solution

$$2^{2k-3}q_2{}^2 = 2s_3u_3\ (a);\ \ p_1 = s_3^2 - u_3^2\ (b);\ \ p_2^2 = s_3^2 + u_3^2\ (c) \qquad (III.6)$$

and from (III.6a) it follows that

$$u_3 = 2^{2k-4}\,\bar{u}_3 \qquad\qquad (III.7)$$

where $\bar{u}_3$ is odd.

We see that the power of 2 in $\bar{u}_3$ is 3 units smaller than in parameter u, defined by Eq. (II.1). If k = 2 then $\bar{u}_3$ becomes odd and equations (III.6b) and (III.6c) are both unsolvable.

Conclusion

Considering the well-known procedure of infinite descent, we have shown that the power of 2 contained in the even parameters of Pythagorean solutions decreases by one with each step of the descent. Therefore on some steps this parameter becomes odd and we arrive at an unsolvable equation. This is the second criterion of descent method.

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